Lagrangian similarity hypothesis applied to diffusion in turbulent shear flow

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The concept suggested by Batchelor that motion of a marked particle in turbulent shear flow may be similar at stations downstream from the point of release is applied to a variety of diffusion data obtained in the laboratory and in the surface layer of the atmosphere. Two types of shear flow parallel to a plane solid boundary are considered. In the first case mean velocity is a linear function of $\log z$ (neutral boundary layer) and in the second case the mean velocity is slightly perturbed from the logarithmic relationship by temperature variation in the z-direction (diabatic boundary layer). Besides the parameters introduced in previous applications of the Lagrangian similarity hypothesis to turbulent diffusion, the ratio of source height to roughness length h/z_0 is shown to be of major importance. Predictions of the variation of maximum ground-level concentration for continuous point and line sources and the variation of plume width for a continuous point source with distance downstream from the source agree with the assorted data remarkably well for a range of length scales extending over three orders-of-magnitude. It is concluded that results from application of the Lagrangian similarity hypothesis are significant for the laboratory modelling of diffusion in the atmospheric surface layer.

1. Introduction

Although no model exists for the turbulent motion in shear flow from which a detailed theory of turbulent diffusion may be constructed, gross characteristics of the concentration field may be predicted through use of similarity arguments. Batchelor (1957) demonstrated the power of similarity reasoning when he applied the hypothesis that turbulent motions of particles in steady, self-preserving, free shear flow possess similarly in the Lagrangian sense. Based on this hypothesis he was able to predict that dispersion and maximum mean concentrations are proportional to certain powers of x for single particle release and continuous particle release. The application of Lagrangian similarity arguments to a turbulent shear flow produced by flow along a solid boundary (boundary-layer flow) in the region where mean velocity varies as the logarithm of wall distance z was later suggested by Batchelor (1959a). For this case where the Eulerian properties of the turbulence structure depend only upon the shear velocity u_*, Batchelor (1959b) and Ellison (1959) determined the way in which the maximum mean concentration at ground level decreases with x at large distances from a continuous point or line source of passive particles released at ground level. When u_* is fixed, the roughness length z_0 appearing in the theory is assumed to affect only the translational velocity of the turbulence field at sufficiently large distances from the boundary with no change in the turbulence structure.

The basic formulation of Batchelor and Ellison is used here to predict the way in which gross characteristics of the concentration field for continuous point and line sources vary with x when the source height is arbitrary and x is not necessarily large. This extended formulation permits use of data from both atmospheric and laboratory diffusion experiments in checking theoretical predictions based on the Lagrangian similarity hypothesis. Laboratory data are provided by a group of studies conducted in a wind tunnel where a tracer gas was diffused (Davar 1961; Poreh 1962; Malhotra 1962) and where heat was diffused (Wieghardt 1948). Atmospheric diffusion data within a neutral surface layer are available from the studies at Porton (Pasquill 1962) and, for approximately neutral conditions, from Project Prairie Grass (Barad 1958).

When vertical temperature gradients greater or smaller than the adiabatic lapse rate exist in the atmospheric surface layer the mean velocity no longer varies linearly with $\log z$ and gross characteristics of the concentration field behave differently than for a neutral atmosphere. Gifford (1962) applied the Lagrangian similarity hypothesis to diffusion in a non-neutral or diabatic surface layer by using a mean velocity distribution modified to include the effects of thermal stratification. The modification used was based on the assumption of Eulerian similarity in which the turbulence characteristics are completely determined by the shear velocity u_* and the stability length L (Monin & Obukhov 1954), and that the roughness length z_0, for fixed L and u_*, affects only the mean velocity of translation. The calculated exponents of x for attenuation of mean concentration obtained by Gifford were in reasonable agreement with experimental values obtained from Project Prairie Grass (Cramer 1957) in which the value of z_0/L ranged from $-0{\cdot}1$ to $0{\cdot}1$.

In the present work only small perturbations of the logarithmic velocity profile are considered where $|z_0/L|$ is of order 10^{-3} or less. Extension of the Lagrangian similarity hypothesis to this case is effected by assuming that the turbulence structure is determined by L and u_* and that z_0 enters explicitly only through its effect upon the mean velocity of translation. The perturbed mean velocity profile is approximated by using the relation

$$du/dz = (u_*/kL)\,(1-e^{-z/L})^{-1}$$

presented by Swinbank (1960) in which terms involving the exponentials are finally expanded in power series. Selected Project Prairie Grass data given by Barad (1958) and data obtained by Malhotra (1962) for diffusion in flow over a heated wind-tunnel floor are compared with the resulting predictions.

All comparisons between theory and experiment made in this work support the hypothesis of Lagrangian similarity.

2. Basic theory

Foundation for the basic theory involved in applying the Lagrangian similarity hypothesis to diffusion in turbulent boundary layers is given by Batchelor (1959*b*) and Ellison (1959). For clarity in extending the basic results and in the interpretation of experimental data, a brief review of the basic theory is presented here. The formulations needed for treating the experimental data are then developed.

2.1. *Review of basic theory*

Only motion of a marked fluid particle or some conserved scalar entity which is carried with the fluid without affecting the fluid motion is considered. The flow considered is a region of the boundary layer where the mean fluid velocity is two-dimensional and is determined by only the shear velocity u_*, excepting for the roughness length z_0 which is a measure of the scale of turbulence where the mean velocity vanishes, i.e. a region where

$$u = u_* f(z/z_0). \tag{1}$$

For such a region of flow the hypothesis may be expressed as follows:

> For a marked particle which is at $z = h$ when $t = 0$, the statistical properties of particle motion at time t depend only upon u_* and $t-t_v$ when t is of order h/u_* or larger, where t_v is a virtual time origin with magnitude of order h/u_*.

A direct result of the hypothesis is that relative to the mean position $(\bar{x}, 0, \bar{z})$, the distribution of particle-displacement probability density for an ensemble of single-particle releases P_{sp} will be similar in shape for t of order h/u_* or larger; thus,

$$P_{sp} = \psi\left(\frac{x-\bar{x}}{\bar{z}}, \frac{y}{\bar{z}}, \frac{z-\bar{z}}{\bar{z}}\right). \tag{2}$$

This form follows from dimensional reasoning since the only length arising from variables in the hypothesis is $u_*(t-t_v)$ which is shown in the next paragraph to be proportional to $\bar{z}$.

For particles released from $z = h$ at $t = 0$, a relationship can be obtained between the mean longitudinal position $\bar{x}$ and the mean vertical positon $\bar{z}$ at any time when t is of order h/u_* or larger. As a consequence of the hypothesis three equations may be written

$$d^2\bar{z}/dt^2 \propto u_*/(t-t_v), \tag{3}$$

$$d^2\bar{x}/dt^2 \propto u_*/(t-t_v) \tag{4}$$

and

$$d^2\bar{y}/dt^2 \propto u_*/(t-t_v). \tag{5}$$

We see that (5) is trivial since the assumed mean flow does not vary in the y-direction; therefore, because of symmetry $d^2\bar{y}/dt^2 = d\bar{y}/dt = 0$, and $\bar{y} = 0$ by proper selection of the origin of co-ordinates. If $d\bar{z}/dt$ is to be finite for all finite time, the constant of proportionality for (3) must be zero and the equation for $\bar{z}$ becomes (for $t \gg h/u_*$)

$$d\bar{z}/dt = bu_*, \tag{6}$$

where b is a new universal constant which we shall call Batchelor's constant. Subject to the condition that $\bar{z} = h$ at $t = 0$, the mean vertical displacement of a particle at time t is given by

$$\bar{z} = bu_* t + h. \tag{7}$$

An integration of (4) gives the rate of change of the mean longitudinal position of a particle as a function of time. However, neglecting longitudinal diffusion, the velocity $d\bar{x}/dt$ may be given with reasonable exactness by the mean fluid velocity at height $z = \bar{z}$ corresponding to $x = \bar{x}$ since the rate of change of $\bar{z}$ is small compared to that for $\bar{x}$; therefore,

$$d\bar{x}/dt = u_* f(\bar{z}/z_0). \tag{8}$$

The time variable t may be replaced with the variable $\bar{z}$, by virtue of (7), to give

$$d\bar{x}/d\bar{z} = (1/b) f(\bar{z}/z_0). \tag{9}$$

Thus, the mean longitudinal position is given by

$$\bar{x} = \frac{1}{b}\int f\left(\frac{\bar{z}}{z_0}\right) d\bar{z} + \text{const.}, \tag{10}$$

and may be obtained from a knowledge of the mean-velocity function f.

The hypothesis is used to obtain information on the concentration field by employing the probability density function in the form given by (2). When Q particles are released from a point instantaneously the concentration χ at (x, y, z) is proportional to the probability density at the same point; therefore,

$$\chi_{\text{instantaneous point source}} = \frac{Q}{\bar{z}^3}\psi\left(\frac{x-\bar{x}}{\bar{z}}, \frac{y}{\bar{z}}, \frac{z-\bar{z}}{\bar{z}}\right). \tag{11}$$

For the continuous point source the mean concentration χ_{cp} is then obtained by integration over all time to give

$$\chi_{cp} = Q_{cp}\int_0^\infty \frac{\psi}{\bar{z}^3}\, dt. \tag{12}$$

Since the function ψ is expected to have a sharp maximum at $x = \bar{x}$, an approximate expression may be obtained for χ_{cp} by changing the variable of integration to $(x-\bar{x})/\bar{z}$ and considering that the contribution at $x = \bar{x}$ dominates the integral. Effecting the change of variable through (7) and (8) gives

$$\chi_{cp} = \frac{Q_{cp}}{bu_*}\int_0^\infty \frac{\psi}{\bar{z}^2}\left[\frac{x-\bar{x}}{\bar{z}} - \frac{1}{b} f\left(\frac{\bar{z}}{z_0}\right)\right]^{-1} d\left(\frac{x-\bar{x}}{\bar{z}}\right), \tag{13}$$

and the maximum ground-level concentration is

$$\chi_{cp}\left(\frac{x-\bar{x}}{z}, 0, 1\right) = \frac{Q_{cp}}{bu_*}\int_0^\infty \bar{z}^{-2}\psi\left(\frac{x-\bar{x}}{\bar{z}}, 0, 1\right)\left[\frac{x-\bar{x}}{\bar{z}} - \frac{1}{b} f\left(\frac{\bar{z}}{z_0}\right)\right]^{-1} d\left(\frac{x-\bar{x}}{\bar{z}}\right). \tag{14}$$

Thus, when ψ has a sharp peak at $x = \bar{x}$,

$$\chi_{cp}(0, 0, 1) \propto \frac{Q_{cp}}{u_*}\frac{1}{\bar{z}^2(x) f(\bar{z}/z_0)}, \tag{15}$$

where $\bar{x}$ and $\bar{z}(x)$ are related by (10).

The fundamental equations resulting from the Lagrangian similarity hypothesis are (10) and (13) while (15) permits calculation of the way in which χ_{cp} varies with distance downstream from the source.

2.2. *Equations for neutral boundary layer, logarithmic velocity distribution*

In this case, expressions for the dependence on x of the maximum ground-level concentration are presented for continuous point and line sources and an expression for plume-width growth with x is given for a continuous point source. The mean velocity distribution is given by

$$u = (u_*/k) \log (z/z_0), \tag{16}$$

where k is the Kármán constant. Particles released at $\bar{z} = h$ when $t = 0$ do not acquire a motion identical with the fluid motion until a time t_v of order h/u^* has elapsed. During this time of relaxation particles travel, in the mean, a distance $\bar{x}_v$ of order $u(h)\,h/u_*$. Insufficient information is available to determine the true form for the relaxation distance $\bar{x}_v$; a crude approximation for $\bar{x}_v$ is $\bar{x}_v = cu(h)\,h/u_*$, when $\bar{z} = h$, and this will be used with the constant of proportionality c taken to be unity. Fortunately, as can be seen from the equations which follow, the uncertainty in the relaxation distance is a defect in the theory only at short distances downstream from the source or where $\bar{x}$ is of order $\bar{x}_v$. The mean trajectory defined by (10) with the approximate condition that $\bar{x} = u(h)\,(h/u_*)$ at $\bar{z} = h$ becomes

$$bk\frac{\bar{x}}{z_0} = \frac{\bar{z}}{z_0}\log\frac{\bar{z}}{z_0} - \left(\frac{\bar{z}-h}{z_0}\right) + (b-1)\frac{h}{z_0}\log\frac{h}{z_0}, \tag{17}$$

or, introducing dimensionless variables $\xi = \bar{x}/z_0$, $\zeta = \bar{z}/z_0$ and $H = h/z_0$,

$$bk\xi = \zeta \log \zeta - (\zeta - H) + (b-1)\, H \log H. \tag{18}$$

The maximum ground-level concentration given by (15) for a continuous point source takes on the form

$$\chi_{cp}(0,0,1) \propto Q_{cp}\, k/u_*\, z_0^2 \zeta^2 \log \zeta \tag{19}$$

and the corresponding expression for the continuous line source is

$$\chi_{cl}(0,0,1) \propto Q_{cl}\, k/u_*\, z_0 \zeta \log \zeta. \tag{20}$$

All the experimental data reported in the literature are used later to determine the power m in an expression of the form $\chi \propto x^m$. Therefore, in comparing theory and experiment, an expression for m must be obtained from (18) and (19) or (20). Since m represents the slope of a tangent to points on the curve where $\log \chi$ is a function of $\log \xi$, the required relationships are

$$m_{cp} = \frac{d(\log \chi_{cp})}{d(\log \xi)} = -(kb\xi)\left(\frac{1 + 2\log \zeta}{\zeta \log^2 \zeta}\right) \tag{21}$$

and

$$m_{cl} = \frac{d(\log \chi_{cl})}{d(\log \xi)} = -(kb\xi)\left(\frac{1 + \log \zeta}{\zeta \log^2 \zeta}\right). \tag{22}$$

If the probability density function does exhibit similarity as expressed by (2), it follows that any measure of the plume width will vary with x in the same manner as does $\bar{z}$ (which is equally true of any measure of the plume height). Let the relationship for plume width Y be

$$Y \propto x^n. \tag{23}$$

We see from (18) that, for the continuous point source,

$$n_{cp} = d(\log \zeta)/d(\log \xi) = bk\xi/\zeta \log \zeta. \tag{24}$$

2.3. *Equations for a non-neutral boundary layer, perturbed logarithmic velocity distribution*

Lagrangian similarity will be assumed for this case, which was also treated by Gifford (1962) without including the parameter H. Only small departures from neutral stability are finally considered, with the result that simple but relatively restricted expressions for m_{cp} and n_{cp} are obtained.

When vertical temperature gradients are caused by heat transfer to or from the solid boundary Monin & Obukhov (1954) have shown by similarity arguments that the effect upon the turbulence structure in the region of constant turbulent shear stress and heat transfer may be measured through a length scale L. This length is defined by

$$L = u_*^3 \Big/ \left(\frac{-kgq}{T\rho c_p}\right),$$

where g is the acceleration due to gravity, T is the mean absolute temperature of the layer and q is the rate of turbulent heat transfer for unit area. For this case ($L \neq 0$), the following hypothesis of Lagrangian similarity is introduced:

> For a marked particle which is at $z = h$ when $t = 0$, the statistical properties of particle motion at time t depend upon u_*, L and $t - t_v$ when t is of order h/u_* or larger (t_v is a virtual time origin with magnitude of order h/u_*). A change in the stability length L produces a change in the scaling length for distributions of the statistical properties but not a change in their shape.

This form of the Lagrangian similarity hypothesis is consistent with the formulation of Gifford (1962) in which the distribution of particle-displacement probability density was taken to be similar when scaled by the reference length $\bar{z}$ alone. In other words, this form is consistent with the assumption that P_{sp} is given by (2) for both the neutral and diabatic boundary layers with only a modification of $\bar{z}$ and $\bar{x}$ resulting from a change in L. Support for this statement of the hypothesis is given by Malhotra's (1962) conclusion that mean concentration distributions are similar in shape for the range of L covered in his experiments. The expression for $\bar{z}$ equivalent to (6) becomes for this case

$$d\bar{z}/dt = bu_* \phi(\bar{z}/L), \tag{25}$$

where Monin (1959) finds the function ϕ to be

$$\phi = \left[1 - \left(k \frac{\partial(u/u_*)}{\partial(\bar{z}/L)}\right)^{-1}\right]^{\frac{1}{4}}. \tag{26}$$

If the form for the mean velocity distribution proposed by Swinbank (1960), viz.

$$k\frac{\partial(u/u_*)}{\partial(z/L)} = (1-e^{-z/L})^{-1}, \tag{27}$$

is accepted, then (25) may be written simply as

$$d\bar{z}/dt = bu_*\, e^{-\frac{1}{4}\alpha\zeta}, \tag{28}$$

where $\alpha = z_0/L$. An integration of (27) and use of (28) gives an equation for the trajectory of mean position of a particle moving in a stratified surface layer. This equation in dimensionless form is

$$\frac{d(bk\xi)}{d\zeta} = \left[\log\left(\frac{1-e^{-\alpha\xi}}{1-e^{-\alpha}}\right) + \alpha(\zeta-1)\right] e^{\frac{1}{4}\alpha\zeta}. \tag{29}$$

Subject to the restrictions that $|\alpha|$ and $|\alpha|\,\zeta \ll 1$ the exponentials in (29) may be approximated by the first two terms of a power-series expansion. Integration of the resulting equation with the condition that $\bar{z} = h$ when $\bar{x} = u(h)\,(h/u_*)$ gives

$$\begin{aligned} bk\xi = {} & (\zeta\log\zeta-\zeta)+\alpha(-\zeta+\tfrac{7}{16}\zeta^2+\tfrac{1}{8}\zeta^2\log\zeta)-(H\log H-H) \\ & -\alpha(-H+\tfrac{7}{16}H^2+\tfrac{1}{8}H^2\log H)+bH[\log H+\alpha(H-1)]. \end{aligned} \tag{30}$$

The expression for maximum ground-level concentration resulting from a continuous point source is obtained from (14) and has the form

$$\chi_{cp} \propto \frac{kQ}{u_* z_0^2}\,\frac{1}{\zeta^2[\log\zeta+\alpha(\zeta-1)]}. \tag{31}$$

Using the same method to obtain m_{cp} as was used to arrive at (21), we find

$$m_{cp} = \frac{-bk\xi}{\zeta}\left\{\frac{2\log\zeta+\alpha(\zeta-1)+\alpha\zeta+1}{[\log\zeta+\alpha(\zeta-1)]\,[\log\zeta+\alpha(\zeta+\frac{1}{4}\zeta\log\zeta-1)]}\right\}. \tag{32}$$

For a continuous line source the exponent m_{cl} is found to be

$$m_{cl} = \frac{-bk\xi}{\zeta}\left\{\frac{\log\zeta+\alpha(\zeta-1)+\alpha\zeta+1}{[\log\zeta+\alpha(\zeta-1)]\,[\log\zeta+\alpha(\zeta+\frac{1}{4}\zeta\log\zeta-1)]}\right\}. \tag{33}$$

The exponent n_{cp} giving the growth rate of the plume width or height for a continuous point source is

$$n_{cp} = \frac{bk\xi}{\zeta[\log\zeta+\alpha(\zeta+\frac{1}{4}\zeta\log\zeta-1)]}. \tag{34}$$

Of course, (34) should give the growth rate of the plume height for a continuous line source also.

3. Diffusion data from experimental studies

Brief descriptions will be presented of the experimental studies in the laboratory and in the atmosphere in which data required to test one or more of the equations for m_{cp}, m_{cl} or n_{cp} were obtained. The essential data from these experiments are tabulated in tables 1 and 2. All experimental values of n_{cp} were obtained from information on plume width.

3.1. *Diffusion data for neutral boundary layers* (*table* 1)

The data of Davar (1961) and Malhotra (1962) were obtained by diffusing ammonia gas in a turbulent boundary layer formed on the smooth floor of a wind-tunnel test section which was 6×6 ft. square and 24 ft. long. By using roughness elements at the beginning of the test section a boundary layer about 3 in. thick was created at the location of the source when the ambient velocity was in the range 6–25 ft./sec. Ammonia gas was introduced through a tube 0·1 in. in diameter penetrating the floor and turned through 90° to emit gas in the direction of mean flow at a maximum elevation of about $\frac{1}{8}$ in.

Poreh (1962) studied the diffusion of ammonia gas from a line source made from a porous strip $\frac{3}{16}$ in. wide placed flush in a smooth floor and orientated at right angles to the mean-flow direction. The work of Poreh was accomplished in a wind-tunnel test section 6×6 ft. square by 80 ft. long with an ambient air-speed range of 9–17 ft./sec. At the source the boundary-layer thickness ranged from 5 to 7 in.

Wieghardt (1948) studied the diffusion of heat created by an electrically heated coil 3 mm in diameter placed in a slot cut into an otherwise smooth floor. To produce the point source, a $1\frac{1}{4}$ in.-long slot with axis in the flow direction was used while the line source was created by a slot cut across the entire width of the tunnel. Dimensions of the tunnel used by Wieghardt were $4\frac{1}{4}$ ft. wide, 1·3 increasing to 2 ft. high and 20 ft. long. The mean air-speed range was 17 to 100 ft./sec.

The exponents given for the field data obtained at Porton represent the mean values of several separate experiments. In each case smoke was diffused into a nearly neutral atmosphere from smoke candles or other smoke generators placed on the ground. The site for the Porton studies was flat grassland. Although no exactly neutral conditions were encountered during the experiments of Project Prairie Grass (Barad 1958), Cramer (1957) estimates the exponents for a point source from the near neutral data. These experiments were realized by releasing sulphur dioxide into the atmosphere from a point source at a height of 30 cm. As at the Porton site, the terrain was flat grassland.

3.2. *Diffusion data for non-neutral boundary layers* (*table* 2)

Laboratory diffusion data for non-neutral flow were obtained by Malhotra (1962) in the same wind tunnel used for the neutral flow excepting that a portion of the floor 6×10 ft. in area was replaced by a heated aluminium plate. Plate surface-ambient air temperature differences up to 200 °F at an air speed of 6 ft./sec were used. The ammonia gas source was placed 2 ft. downstream from the upstream edge of the heated plate with this edge being 4 ft. downstream from the origin of the momentum boundary layer. This produced a momentum boundary-layer thickness of about 3·5 in. and a temperature boundary-layer thickness of about 2 in. at the source.

Only two experiments of the Project Prairie Grass series are examined in detail. These particular experiments (23 and 57) were selected because the inversion and lapse respectively were weak and because no large-scale transverse wind fluctuations occurred during the experiments to produce multiple maxima in the transverse concentration distribution.

Experiment	x (ft.)	z_0 (ft.)	h (ft.)	L	$kb\xi$	H	α	$-m_{cp}$	$-m_{cl}$	n_{cp}
1. Laboratory, point source										
Davar 1961	1·5	$9{\cdot}9\times10^{-5}$	$9{\cdot}38\times10^{-3}$	∞	621	95	0	1·20	—	—
	4·5	10·5	9·38	∞	1750	87	0	1·47	—	—
Malhotra 1962	4·5	2·5	5·20	∞	7380	208	0	1·47	—	—
	2·5	2·5	5·20	∞	4100	208	0	—	—	0·60
	2·5	10·5	9·38	∞	975	87	0	—	—	0·60
Wieghardt 1948	1·03	3·7	—	∞	1140	—	0	1·42	—	0·67
	1·03	2·0	—	∞	2130	—	0	1·42	—	0·67
2. Laboratory, line source										
Malhotra 1962	4·5	$2{\cdot}5\times10^{-5}$	$5{\cdot}20\times10^{-3}$	∞	7380	208	0	—	0·80	—
Poreh 1962	7·5	6·2	—	∞	4350	—	0	—	0·90	—
	7·5	4·8	—	∞	6400	—	0	—	0·90	—
	7·5	3·5	—	∞	8700	—	0	—	0·90	—
Wieghardt 1948	1·23	3·8	—	∞	1320	—	0	—	0·90	—
	1·23	2·04	—	∞	2470	—	0	—	0·90	—
3. Field, point source										
Porton (Pasquill 1962)	1640	$9{\cdot}85\times10^{-2}$	$5^*\ \times10^{-1}$	∞	685	5	0	1·76	—	—
	656	9·85	5	∞	273	5	0	—	—	0·74
Prairie Grass (Cramer 1957)	1976	3·28	9·85	∞	2460	30	0	1·8	—	0·80
4. Field, line source										
Porton (Pasquill 1962)	1640	$9{\cdot}85\times10^{-2}$	$5^*\ \times10^{-1}$	∞	685	5	0	—	0·9–1·0	—

* Estimated with the assistance of Dr Pasquill.

TABLE 1. Data on diffusion in neutral boundary layers.

Experiment	x (ft.)	z_0 (ft.)	h (ft.)	L (ft.)	$kb\xi$	H	α	$-m_{cp}$ Exp.	$-m_{cp}$ Calc.	n_{cp} Exp.	n_{cp} Calc.
1. Laboratory, point source											
Malhotra 1962	4	$1{\cdot}08\times10^{-4}$	$9{\cdot}38\times10^{-3}$	−0·34	1440	87	$-3{\cdot}2\times10^{-4}$	1·60	1·58	0·69	0·71
	4	$8{\cdot}3\times10^{-5}$	$9{\cdot}38\times10^{-3}$	−0·92	1950	113	$-8{\cdot}9\times10^{-5}$	1·61	1·51	0·68	0·70
	4	$1{\cdot}53\times10^{-4}$	$9{\cdot}38\times10^{-3}$	−0·09	1050	61	$-1{\cdot}7\times10^{-3}$	1·84	1·75	0·73	0·80
2. Field, point source											
Prairie Grass (Barad 1958)											
Expt. no. 23	246	$3{\cdot}28\times10^{-2}$	$9{\cdot}85\times10^{-1}$	19·7	307	30	$1{\cdot}67\times10^{-3}$	1·48	—	—	—
	492	$3{\cdot}28\times10^{-2}$	$9{\cdot}85\times10^{-1}$	19·7	615	30	$1{\cdot}67\times10^{-3}$	1·56	—	—	—
	988	$3{\cdot}28\times10^{-2}$	$9{\cdot}85\times10^{-1}$	19·7	1230	30	$1{\cdot}67\times10^{-3}$	1·64	—	—	—
	1976	$3{\cdot}28\times10^{-2}$	$9{\cdot}85\times10^{-1}$	19·7	2460	30	$1{\cdot}67\times10^{-3}$	1·70	—	—	—
Expt. no. 57	246	$3{\cdot}28\times10^{-2}$	$9{\cdot}85\times10^{-1}$	−39·4	307	30	$-8{\cdot}33\times10^{-4}$	1·59	—	—	—
	492	$3{\cdot}28\times10^{-2}$	$9{\cdot}85\times10^{-1}$	−39·4	615	30	$-8{\cdot}33\times10^{-4}$	1·69	—	—	—
	988	$3{\cdot}28\times10^{-2}$	$9{\cdot}85\times10^{-1}$	−39·4	1230	30	$-8{\cdot}33\times10^{-4}$	1·81	—	—	—
	1976	$3{\cdot}28\times10^{-2}$	$9{\cdot}85\times10^{-1}$	−39·4	2460	30	$-8{\cdot}33\times10^{-4}$	1·89	—	—	—

TABLE 2. Data on diffusion in mildly diabatic boundary layers.

3.3. *Calculation of parameters*

For the laboratory experiments, in which the boundaries were all smooth, the value of z_0 was estimated by using the mean velocity function

$$u/u_* = k^{-1}\log(zu_*/\nu)+4\cdot 9.$$

This gives the result $z_0 = 0\cdot 141(\nu/u_*)$. The value of the shear velocity u_* was taken as the mean value over the distance x up to where the exponent of x was measured and was calculated either by differentiation of the momentum thickness θ or by using the Schultz–Grunow drag formula $(2/C_f)^{\frac{1}{2}} = 6\cdot 30\log(u_{amb}\theta/\nu)+2\cdot 40$. The stability length L was obtained from the basic definition for Malhotra's data since all the necessary quantities were known. For the Prairie Grass data L was obtained from the integral of (27) using the mean velocity at three different elevations.

In all cases the exponents m_{cp}, m_{cl} and n_{cp} were obtained by measuring the slope of a tangent drawn to curves constructed by plotting the logarithm of the appropriate variable as a function of $\log x$. In all cases x was restricted so that $\lambda/\delta \leqslant 0\cdot 40$. In this ratio δ is the boundary-layer thickness and λ is a characteristic length of the concentration field, the height z where the mean concentration is one-half the maximum. Poreh found that for a line source the concentration profiles are similar for $\lambda/\delta \leqslant 0\cdot 40$ but then gradually change form until $\lambda/\delta = 0\cdot 64$, when a new similarity profile is attained. Since the similarity hypothesis is formulated only for the inner part of the boundary layer, only the first similarity region is strictly within the scope of the analysis.

In numerical calculations, the value of Kármán's constant k has been taken as 0·41 and the value of b (Batchelor's constant) has been taken as 0·1. The value of 0·1 for b gives good agreement with the data (and, as can be seen from figure 1, a change in b from 0·1 to 0·2 or larger gives poor agreement) but should be considered only as a rough approximation until more diffusion data are available to determine the true value. Batchelor (1959*b*) estimated b to be about 0·1 or 0·2. A rough estimate of Batchelor's constant may be obtained by multiplying the approximate maximum vertical plume velocity of $0\cdot 75u_*$ reported by Monin (1959) by the ratio of elevation at mean concentration to elevation at 0·01 of maximum concentration (outer edge of plume). Using the exponential function for the vertical concentration distribution given by Calder (1952), this ratio of elevations is 4·82, which gives a value for b of 0·15. Ellison (1959) found that b would be equal to k and thus have a value of about 0·4 provided that a K-theory of diffusion is valid and the vertical diffusion coefficients for mass and momentum are the same; the data described here does not support such a large value of k.

4. Discussion

Data given in tables 1 and 2 obtained from the studies briefly described in the preceding section may be used to determine the validity of results obtained from the hypothesis of Lagrangian similarity. Figures 1 to 4 show both the experimental data and selected theoretical curves to facilitate comparison. For an *aerodynamically smooth surface* the curve where $H = 75$ corresponds to h equal to the laminar sublayer thickness and $H = 225$ corresponds to an h where transition to the logarithmic profile has been completed. The curves for H equal to

30 and 100 correspond closely with the value of H for the Prairie Grass data and certain wind-tunnel data, respectively. The degree of agreement between theory and data is sufficiently good to justify use of the Lagrangian similarity hypothesis as the basis of diffusion modelling in the atmospheric surface layer.

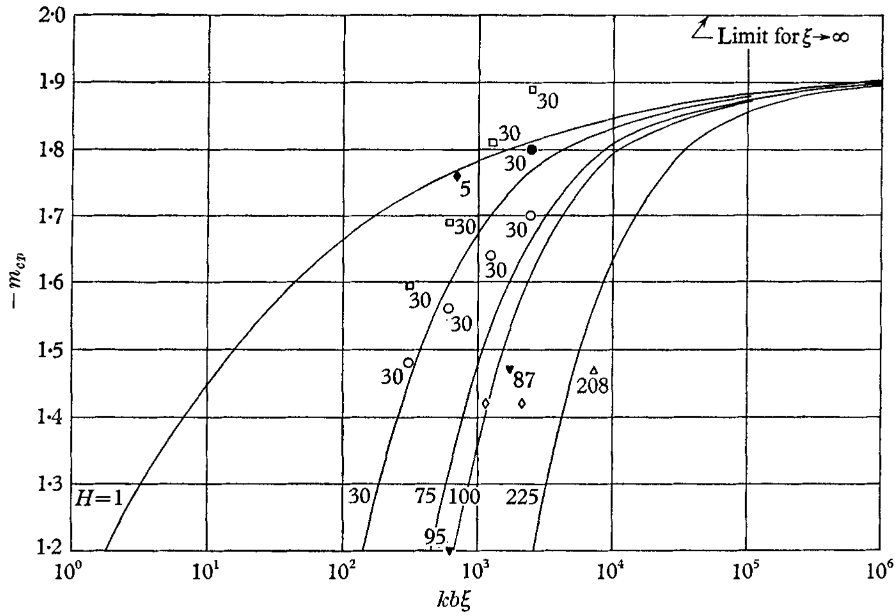

FIGURE 1. The exponent-of-distance m_{cp} for attenuation of maximum ground-level concentration as a function of distance and source height resulting from a point source in a neutral boundary layer: ——, equations (18) and (21); ▼, Davar (1961); △, Malhotra (1962); ◇, Wieghardt (1948); ●, mean Prairie Grass data, $\alpha = 0$ (Cramer 1957); ◆, mean Porton data, $\alpha = 0$ (Pasquill 1962); ○, Prairie Grass experiment no. 23, $\alpha = 1{\cdot}67 \times 10^{-3}$ (Barad 1958); □, Prairie Grass experiment no. 57, $\alpha = -8{\cdot}33 \times 10^{-4}$ (Barad 1958). Numbers opposite data points give values of H.

4.1. *Comparison of theory and experiment*

In figure 1 all the values of m_{cp} for aerodynamically smooth boundaries (wind-tunnel experiments) correspond to theoretical values of $H > 75$. Where the value of H is known from the experimental conditions, the exponent m_{cp} from experiment is within 10 % of the corresponding theoretical value. For the data of Wieghardt in which H is not explicitly known, the relationship between the experimental points and the theoretical curves indicate that the apparent source height for a source embedded in a smooth boundary is between the height of the laminar sublayer and the height corresponding to where the logarithmic velocity distribution is attained. The mean Porton and Prairie Grass data for neutral conditions shown in figure 1 are within about 2 % of the corresponding theoretical values. Project Prairie Grass data for mildly non-neutral surface layers are also shown in figure 1 to emphasize the importance of the parameter H in the present theory. As is evident, the mild lapse and mild inversion produce values of m_{cp} which diverge with increasing distance ξ, above and below respectively, from the neutral curve for $H = 30$.

Figure 2 gives the available experimental data and theoretical curves for attenuation of maximum ground-level concentration for a continuous line source in neutral boundary layers in terms of m_{cl}. The wind-tunnel data of known H due to Malhotra was obtained by integration of his point-source concentrations and is about 5 % lower than the corresponding theoretical value. Wind-tunnel data of Poreh with gas emitted from a smooth porous line source is consistent with the idea of the effective source-height parameter for such sources being in the range $75 < H < 225$. The progression of points from left to right represent

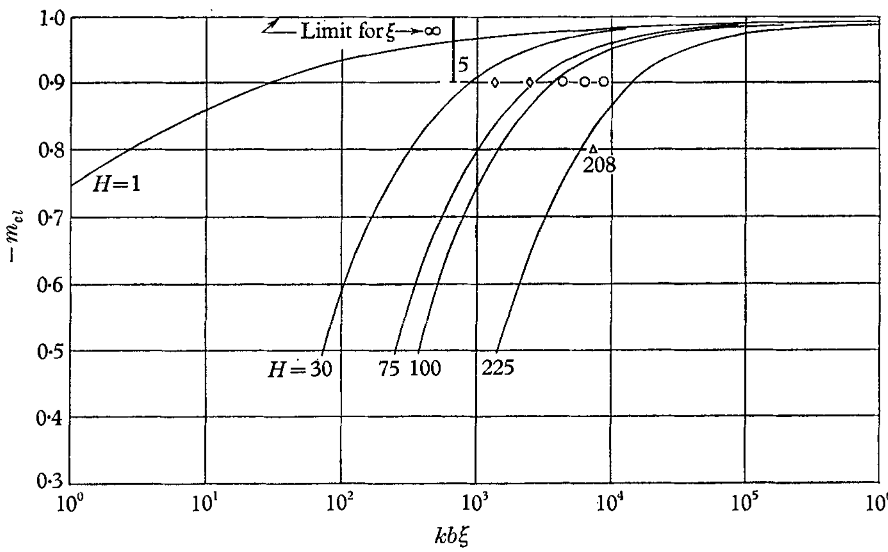

FIGURE 2. The exponent-of-distance m_{cl} for attenuation of maximum ground-level concentration as a function of distance and source height resulting from a line source in a neutral boundary layer: ——, equations (18) and (22); △, integrated point-source data, Malhotra (1962); ○, Poreh (1962); ◇, Wieghardt (1948); I, Porton data (Pasquill 1962). Numbers opposite data points give values of H.

increasing mean ambient velocity and consequently decreasing values of z_0. Since the data of Wieghardt correspond to values of $H < 75$ for this case, it is concluded that the transverse boundary slot containing the source produced a large-scale disturbance making the effective z_0 (local turbulence scale) larger than that calculated by considering the boundary to be smooth. Field data obtained at Porton give a range of m_{cl} with a mean value very near the predicted value of about $-0{\cdot}95$. The scatter of m_{cl} at this site can easily be accounted for through varying roughness and small departures from neutral conditions.

Both average values of n_{cp} giving the rate of plume-width growth for Porton and Prairie Grass data under neutral conditions are in good agreement with the theoretical values shown in figure 3. Values of n_{cp} for the neutral wind-tunnel data and the corresponding theoretical values are also in satisfactory agreement.

The data and theoretical curves for m_{cp} when small departures from neutral conditions exist are given in figure 4 and table 2. The two sets of Project Prairie Grass data are in fair agreement with the curves for $H = 30$ and $\alpha = \pm 0{\cdot}001$.

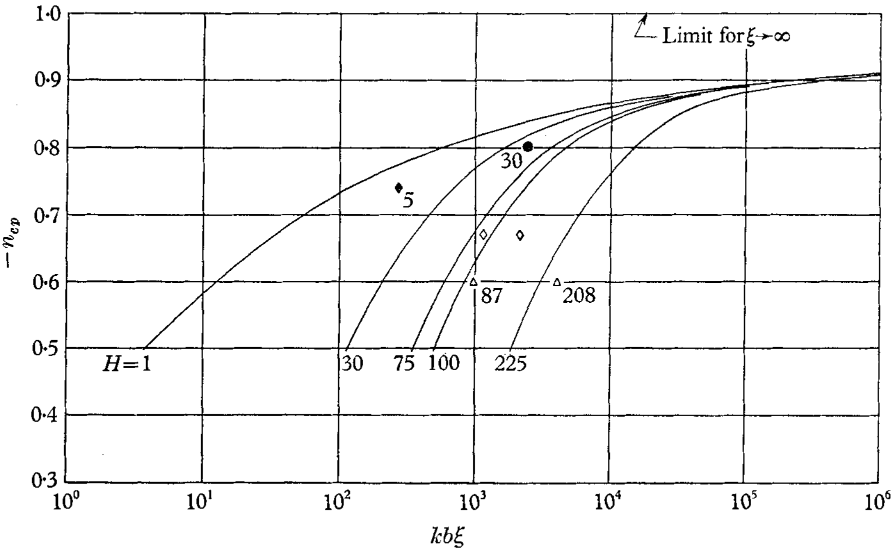

FIGURE 3. The exponent-of-distance n_{cp} for plume-width growth as a function of distance and source height resulting from a point source in a neutral boundary layer: ——, equations (18) and (24); △, Malhotra (1962); ◇, Wieghardt (1948); ●, mean Prairie Grass data, $\alpha = 0$ (Cramer 1957); ◆, mean Porton data, $\alpha = 0$ (Pasquill 1962). Numbers opposite data points give values of H.

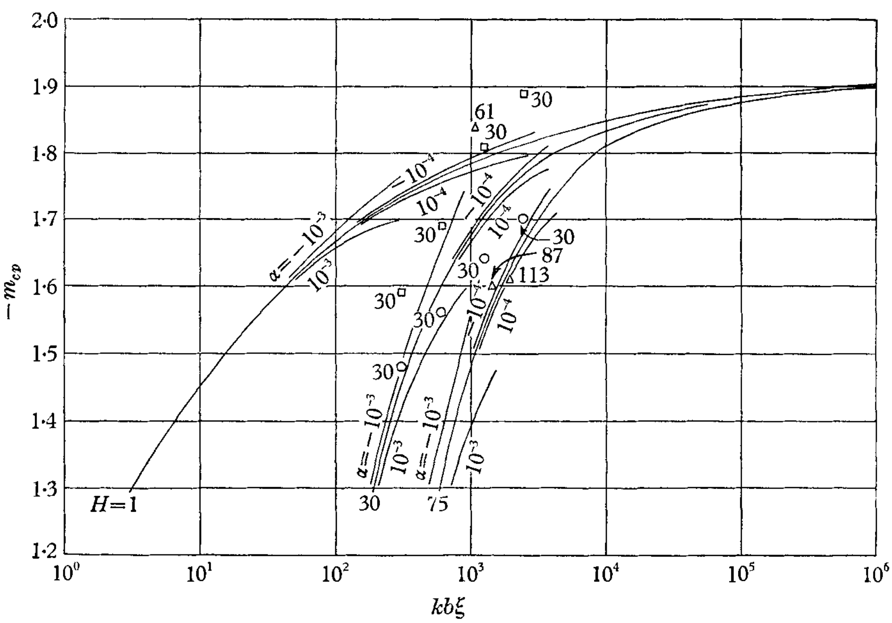

FIGURE 4. The exponent-of-distance m_{cp} for attenuation of maximum ground-level concentration as a function of distance, source height and stability resulting from a point source in a diabatic boundary layer: ——, equations (30) and (32); △, Malhotra (1962); ○, Prairie Grass experiment no. 23, $\alpha = 1{\cdot}67 \times 10^{-3}$ (Barad 1958); □, Prairie Grass experiment no. 57, $\alpha = -8{\cdot}33 \times 10^{-4}$ (Barad 1958). Numbers opposite data points give values of H.

Gifford, in treating average values of m_{cp} over certain ranges of the stability length covering the entire range of atmospheric conditions encountered during the Prairie Grass experiments, found good agreement with his theoretical predictions. Although Gifford included the effect of large departures from neutral conditions by using the mean velocity distribution functions of Monin (1959), he did not consider the important parameter H. Calculated and experimental values of m_{cp} and n_{cp} corresponding to the conditions created over a heated plate in the studies of Malhotra are given in table 2. Fair agreement is indicated with $\pm 8\,\%$ differences. It should be noted that the calculated values of m_{cp} are consistently lower and that the values of n_{cp} are consistently higher than the experimental values and the differences generally increase with increasing instability. This appears to be caused by small secondary circulations producing upward flow above the centre of the heated boundary where the diffusion of mass was centred.

4.2. *Significance of findings for modelling*

Diffusion in the boundary layer of a wind-tunnel model of the atmospheric surface layer will, according to (30), (32) and (34), be similar to the prototype if the parameters H and α are the same for both cases. Inoue (1959) reached the same conclusion by requiring that the angle of diffusion and a dimensionless diffusion length $U\tau/h$ (τ is the Lagrangian time-scale for motion in the direction of mean velicity U) be the same for both model and prototype. Of course, the modelling can be accomplished only if the wind-tunnel boundary layer is sufficiently thick to ensure that $\lambda/\delta \leqslant 0{\cdot}4$ over the ranges of ξ covering the model.

An even more important consequence of the agreement between results of the Lagrangian similarity hypothesis and data from field and laboratory is the implied similarity of the turbulence structures. This means that in properly designed laboratory experiments measurements of turbulence structure for controlled stability and roughness will yield information applicable to the atmospheric surface layer.

5. Conclusions

Examination of the data and analytical results presented here strongly support the Lagrangian similarity hypothesis. This simple but powerful concept affords a rational basis for describing the gross characteristics of a diffusion field within a turbulent boundary layer. The use of z_0 as a reference scale of turbulence successfully accounts for differences in diffusion rates when diffusion takes place on scales varying from those encountered in the laboratory to those existing in the atmospheric surface layer. Further study of diffusion rates for a wide range of z_0 and H, such as can be accomplished in a wind tunnel using boundaries with fixed or flexible roughness elements accompanied by heating or cooling of the boundary would be particularly illuminating. In applying the existing formulation of the Lagrangian similarity hypothesis to diffusion in wind-tunnel boundary layers the plume of diffusing mass or heat should be well within the boundary layer (i.e. $\lambda/\delta \leqslant 0{\cdot}40$). On the other hand, the foregoing analytical results do not apply immediately downstream from the source for distances smaller than $u(h)\,(h/u_*)$.

The analysis states that two fields of diffusion within the inner region of a turbulent boundary layer will be similar if H and α for one field are equal to H and α respectively for the other field. This not only gives a basis for modelling practical cases of diffusion in the atmospheric surface layer but also provides a means by which laboratory measurements of basic turbulence structure may be applied to the atmosphere.

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